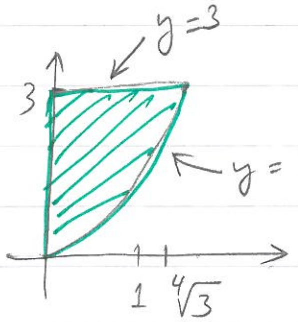
Lektion 17

6.6

a)



$$\iint_D x^3 y \, dx\, dy =$$

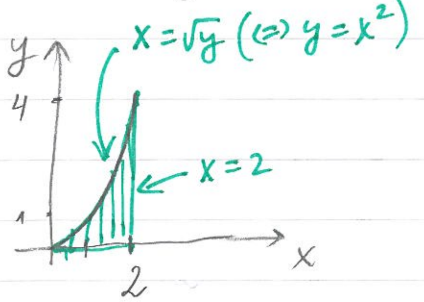
$$= \int_0^{\sqrt[4]{3}} \left(\int_{x^4}^{3} x^3 y \, dy \right) dx =$$

$$= \int_0^{\sqrt[4]{3}} \left[\frac{x^3 y^2}{2} \right]_{y=x^4}^{y=3} dx =$$

$$= \int_0^{\sqrt[4]{3}} \left(\frac{9x^3}{2} - \frac{x^{11}}{2} \right) dx = \left[\frac{9x^4}{8} - \frac{x^{12}}{24} \right]_{x=0}^{x=\sqrt[4]{3}} =$$

$$= \frac{9}{8}\left(\sqrt[4]{3}\right)^4 - \frac{\left(\sqrt[4]{3}\right)^{4\cdot 3}}{24} = \frac{27}{8} - \frac{\cancel{27}\,9}{\cancel{24}\,8} = \frac{18}{8} = \underline{\underline{\frac{9}{4}}}$$

b) $\sqrt{y} \leq x \leq 2$ är ett område mellan kurvorna $x = \sqrt{y}$ och $x = 2$, $x \geq 0$. Den kan också skrivas som



$$D = \{ 0 \leq x \leq 2, \quad 0 \leq y \leq x^2 \},$$

så

$$\iint_D x \cos\sqrt{y} \, dx\, dy =$$

$$= \int_0^2 \left(\int_0^{x^2} x \cos\sqrt{y} \, dy \right) dx = \otimes$$

Observera att

$$\int \cos\sqrt{y} \, dy = \left[\begin{matrix} \sqrt{y} = t \\ y = t^2 \end{matrix} \Rightarrow dy = 2t\,dt \right] = 2\int \cos t \cdot t \, dt =$$

$$= 2\sin t \cdot t - 2\int \sin t \, dt = 2t\sin t + 2\cos t =$$

$$= 2\sqrt{y}\sin\sqrt{y} + 2\cos\sqrt{y} \implies$$

$$\otimes = \int_0^2 \left(x\left[2\sqrt{y}\sin\sqrt{y} + 2\cos\sqrt{y}\right]_{y=0}^{y=x^2} \right) dx =$$

$$= \int_0^2 x\left[2x\sin x + 2\cos x - 2\right] dx =$$

$$= \int_0^2 \left(2x^2\sin x + 2x\cos x - 2x\right) dx =$$

$$= \left[-2x^2\cos x\right]_{x=0}^{x=2} + 4\int_0^2 x\cos x \, dx + 2\int_0^2 x\cos x - \left[x^2\right]_{x=0}^{x=2} =$$

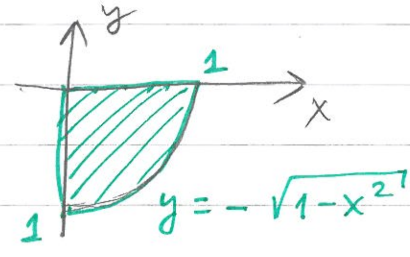
$$= -8\cos 2 + 6\underbrace{\int_0^2 x\cos x \, dx}_{\text{se ovan}} - 4 =$$

$$= -8\cos 2 - 4 + 6\left[x\sin x + \cos x\right]_{x=0}^{x=2} =$$

$$= -8\cos 2 - 4 + 12\sin 2 + 6\cos 2 - 6 =$$

$$= 12\sin 2 - 2\cos 2 - 10.$$

c)



$$D = \{0 \le x \le 1, \ -\sqrt{1-x^2} \le y \le 0\}.$$

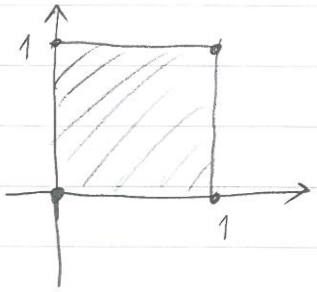
$$\iint_D xy \, dx\,dy = \int_0^1 \left(\int_{-\sqrt{1-x^2}}^{0} xy \, dy \right) dx =$$

$$= \int_0^1 \left[\frac{xy^2}{2}\right]_{y=-\sqrt{1-x^2}}^{y=0} dx = \int_0^1 \left(0 - \frac{x(1-x^2)}{2}\right) dx =$$

$$= \int_0^1 \left(-\frac{x}{2} + \frac{x^3}{2} \right) dx = \left[-\frac{x^2}{4} + \frac{x^4}{8} \right]_{x=0}^{x=1} =$$

$$= -\frac{1}{4} + \frac{1}{8} = \frac{-2+1}{8} = -\frac{1}{8}.$$

6.7

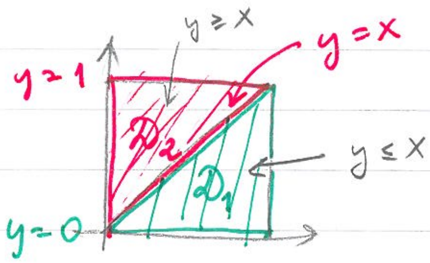


$$\mathcal{D} = \{ 0 \le x \le 1,\ 0 \le y \le 1 \}.$$

a) $$|x-y| = \begin{cases} x-y, & \text{då } x-y \ge 0 \\ -(x-y), & \text{då } x-y \le 0 \end{cases} \iff$$

$$|x-y| = \begin{cases} x-y & \text{då } y \le x \\ -x+y & \text{då } y \ge x \end{cases}.$$

Vi ser att det är rimligt att splitra $\mathcal{D}$ i två områden:



$\mathcal{D}_1 = \{0 \le x \le 1,\ 0 \le y \le x\}$, där

$|x-y| = x-y$ och

$\mathcal{D}_2 = \{0 \le x \le 1,\ x \le y \le 1\}$ där

$|x-y| = y-x$.

I så fall

$$\iint_{\mathcal{D}} |x-y| \, dx\,dy = \iint_{\mathcal{D}_1} (x-y)\, dx\,dy + \iint_{\mathcal{D}_2} (y-x)\, dx\,dy,$$

där $$\iint_{\mathcal{D}_1} (x-y)\,dx\,dy = \int_0^1 \left(\int_0^x (x-y)\,dy \right) dx =$$

$$= \int_0^1 \left[xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx = \int_0^1 \left(x^2 - \frac{x^2}{2} \right) dx = \int_0^1 \frac{x^2}{2}\, dx =$$

$$= \left[\frac{x^3}{6} \right]_{x=0}^{x=1} = \frac{1}{6}$$

och

$$\iint_{D_2} (y-x)\, dx\, dy = \int_0^1 \left(\int_x^1 (y-x)\, dy \right) dx =$$

$$= \int_0^1 \left[-xy + \frac{y^2}{2} \right]_{y=x}^{y=1} dx = \int_0^1 \left(-x + \frac{1}{2} + x^2 - \frac{x^2}{2} \right) dx =$$

$$= \int_0^1 \left(-x + \frac{1}{2} + \frac{x^2}{2} \right) dx = \left[-\frac{x^2}{2} + \frac{1}{2}x + \frac{x^3}{6} \right]_{x=0}^{x=1} =$$

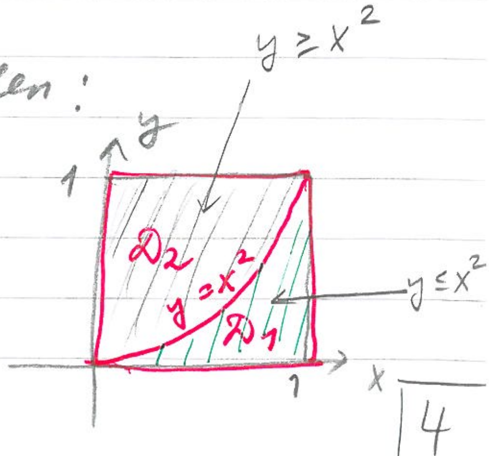
$$= -\frac{1}{2} + \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.$$

$$\iint_D |x-y|\, dx\, dy = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

b) Observera att

$$\max(x^2, y) = \begin{cases} x^2, & \text{då } y \le x^2 \\ y, & \text{då } y \ge x^2 \end{cases}.$$

Vi splitrar igen D i två områden:



$D_1 = \{0 \le x \le 1;\ 0 \le y \le x^2\}$, där
$\max(x^2, y) = x^2$ och

$D_2 = \{0 \le x \le 1;\ x^2 \le y \le 1\}$ där
$\max(x^2, y) = y$.

Så

$$\iint_{D} \max(x^2, y)\,dx\,dy = \iint_{D_1} x^2\,dx\,dy + \iint_{D_2} y\,dx\,dy, \text{ där}$$

$$\iint_{D_1} x^2\,dx\,dy = \int_0^1 \left(\int_0^{x^2} x^2\,dy \right) dx = \int_0^1 x^2 \left[y \right]_{y=0}^{y=x^2} dx =$$

$$= \int_0^1 x^4\,dx = \left[\frac{x^5}{5} \right]_{x=0}^{x=1} = \frac{1}{5}.$$

$$\iint_{D_2} y\,dx\,dy = \int_0^1 \left(\int_{x^2}^1 y\,dy \right) dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^{y=1} dx =$$

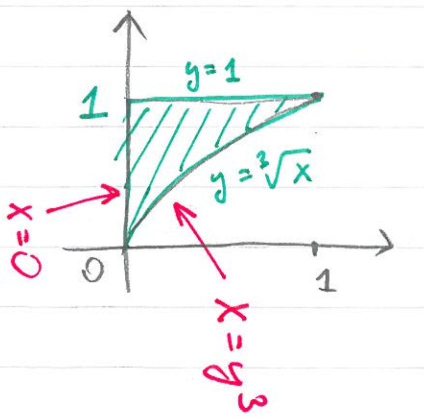
$$= \int_0^1 \left(\frac{1}{2} - \frac{x^4}{2} \right) dx = \left[\frac{x}{2} - \frac{x^5}{10} \right]_{x=0}^{x=1} =$$

$$= \frac{1}{2} - \frac{1}{10} = \frac{5-1}{10} = \frac{4}{10} = \frac{2}{5}.$$

Totalt: $\iint_{D} \max(x^2, y)\,dx\,dy = \frac{3}{5}$.

6.9

a) $$\int_0^1 \left(\int_{\sqrt[3]{x}}^1 \frac{dy}{\sqrt{1+y^8}} \right) dx = \iint_{D} \frac{dx\,dy}{\sqrt{1+y^8}} \quad \text{där}$$



$D = \{ 0 \le x \le 1,\ \sqrt[3]{x} \le y \le 1 \}$.

Eftersom $\int \frac{dy}{\sqrt{1+y^8}}$ verkar vara svår att beräkna byter vi integrationsordning.

Vi skriver istället

$\mathcal{D} = \{ 0 \le y \le 1, \ 0 \le x \le y^3 \}$ så integrallen blir

$$\int_0^1 \left(\int_0^{y^3} \frac{1}{\sqrt{1+y^8}} \, dx \right) dy = \int_0^1 \frac{1}{\sqrt{1+y^8}} \Big[x \Big]_{x=0}^{x=y^3} dy =$$

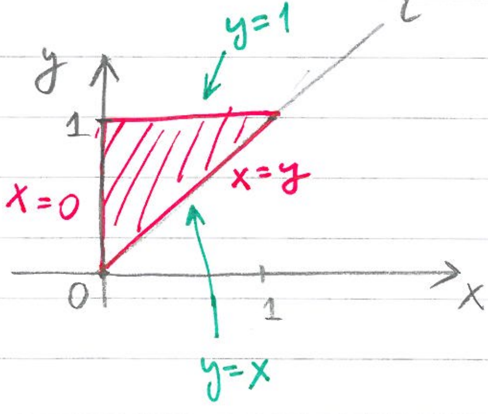
$$= \int_0^1 \frac{y^3}{\sqrt{1+y^8}} \, dy = \begin{bmatrix} y^4 = t & 0 \le y \le 1 \\ dt = 4y^3 dy & 0 \le t \le 1 \end{bmatrix} =$$

$$= \frac{1}{4} \int_0^1 \frac{4y^3 dy}{\sqrt{1+(y^4)^2}} = \frac{1}{4} \int_0^1 \frac{dt}{\sqrt{1+t^2}} = \left[\frac{1}{4} \ln \left| t + \sqrt{1+t^2} \right| \right]_{t=0}^{t=1} =$$

$$= \frac{1}{4} \ln (1+\sqrt{2}) - \frac{1}{4} \ln 1 = \frac{\ln(1+\sqrt{2})}{4}$$

b) $$\int_0^1 \left(\int_0^y \frac{y \, dx}{(4-x^2-y^2)^{3/2}} \right) dy = \iint_{\mathcal{D}} \frac{y \, dx dy}{(4-x^2-y^2)^{3/2}}$$

där $\mathcal{D} = \{ 0 \le y \le 1, \ 0 \le x \le y \}$.



$\int_0^y \frac{y \, dx}{(4-x^2-y^2)^{3/2}}$ verkar

vara svår att beräkna ⇒

byter integrationsordning.

Vi skriver $\mathcal{D} = \{ 0 \le x \le 1, \ x \le y \le 1 \}$ så

integralen blir

$$\int_0^1 \left(\int_x^1 \frac{y}{(4-x^2-y^2)^{3/2}} \, dy \right) dx = \left[\begin{array}{l} \left((4-x^2-y^2)^{-1/2} \right)'_y = \\ = \left(-\frac{1}{2}\right)(-2y)(4-x^2-y^2)^{-\frac{3}{2}} \\ \Rightarrow \int \frac{y \, dy}{(4-x^2-y^2)^{3/2}} = \frac{1}{\sqrt{4-x^2-y^2}} \end{array} \right] =$$

$$= \int_0^1 \left[\frac{1}{\sqrt{4-x^2-y^2}} \right]_{y=x}^{y=1} dx = \int_0^1 \left(\frac{1}{\sqrt{3-x^2}} - \frac{1}{\sqrt{4-2x^2}} \right) dx =$$

$$= \frac{1}{\sqrt{3}} \int_0^1 \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}} dx - \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} =$$

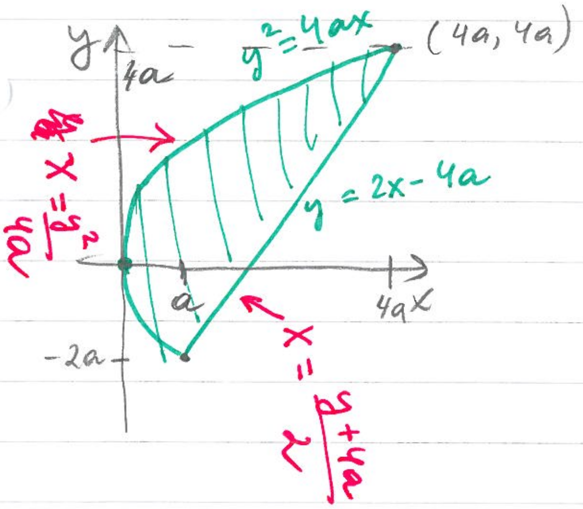
$$= \int_0^1 \frac{\frac{1}{\sqrt{3}} dx}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}} - \frac{1}{\sqrt{2}} \int_0^1 \frac{\frac{1}{\sqrt{2}} dx}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} =$$

$$= \left[\arcsin \frac{x}{\sqrt{3}} \right]_{x=0}^{x=1} - \frac{1}{\sqrt{2}} \left[\arcsin \frac{x}{\sqrt{2}} \right]_{x=0}^{x=1} =$$

$$= \arcsin \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}} \cdot \frac{\pi}{4} = \arcsin \frac{1}{\sqrt{3}} - \frac{\pi\sqrt{2}}{8}.$$

Extra

6.8



Linjen $y = 2x - 4a$ och parabeln $y^2 = 4ax$ skär varandra i punkterna som satisfierar systemet

$$\begin{cases} y = 2x - 4a \\ y^2 = 4ax \end{cases} \Leftrightarrow \begin{cases} (2x-4a)^2 = 4ax \\ y = 2x - 4a \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \cancel{4}x^2 - \cancel{16}^{4} xa + \cancel{16}^{4} a^2 = \cancel{4}ax \\ y = 2x - 4a \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^2 - 5ax + 4a^2 = 0 \\ y = 2x - 4a \end{cases} \Leftrightarrow \begin{cases} (x-a)(x-4a) = 0 \\ y = 2x - 4a \end{cases} \Rightarrow$$

$\Rightarrow$ Skärningspunkterna är $(a, -2a)$ och $(4a, 4a)$.

Det är bekvamt att skriva

$$D = \left\{ -2a \le y \le 4a, \quad \frac{y^2}{4a} \le x \le \frac{y+4a}{2} \right\} \Rightarrow$$

$$\iint_D xy \, dx\,dy = \int_{-2a}^{4a} \left(\int_{\frac{y^2}{4a}}^{\frac{y+4a}{2}} xy \, dx \right) dy =$$

$$= \int_{-2a}^{4a} y \left[\frac{x^2}{2} \right]_{x=\frac{y^2}{4a}}^{x=\frac{y+4a}{2}} dx =$$

$$= \int_{-2a}^{4a} y \left(\frac{(y+4a)^2}{8} - \frac{y^4}{32a^2} \right) dy =$$

$$= \int_{-2a}^{4a} y \left(\frac{y^2 + 8ay + 16a^2}{8} - \frac{y^4}{32a^2} \right) dy =$$

$$= \int_{-2a}^{4a} \left(\frac{y^3}{8} + ay^2 + 2a^2 y - \frac{y^5}{32a^2} \right) dy =$$

$$= \left[\frac{y^4}{32} + \frac{ay^3}{3} + a^2y^2 - \frac{y^6}{6 \cdot 32a^2} \right]_{y=-2a}^{y=4a} =$$

$$= \left(\frac{256a^4}{32} - \frac{16a^4}{32} \right) + \frac{64a^4}{3} + \frac{8a^4}{3} + 16a^4 - 4a^4$$

$$- \frac{4^6 a^4}{6 \cdot 32} + \frac{64a^4}{6 \cdot 32} = 8a^4 - \frac{a^4}{2} + \frac{72a^4}{3} + 12a^4$$

$$- \frac{64 \cdot \cancel{64} \cdot a^4}{3 \cdot \cancel{64}} + \frac{\cancel{64}a^4}{3 \cdot \cancel{64}} = 8a^4 - \frac{a^4}{2} + 24a^4 + 12a^4$$

$$- \frac{\overset{21}{\cancel{63}}}{\cancel{3}} a^4 = 23 - \frac{1}{2} = \frac{46-1}{2} = \frac{45}{2}$$