# Lektion 6: standarta gränsvärden (forts)

$\boxed{B3}$ $\ln x << x^{\alpha} << \cancel{\ln}\ a^x$ där $\alpha > 0$, $a > 1$ när $x \to \infty$

**31** a) $\lim\limits_{x\to 0} \dfrac{e^{2x} - 2e^x + 1}{x^2} =$

$\Rightarrow \dfrac{\ln x}{x^{\alpha}} \to 0$ då $x \to \infty$, $\alpha > 0$

$\dfrac{x^{\alpha}}{a^x} \to 0$ då $x \to \infty$, $a > 1$

$$= \lim_{x\to 0} \frac{(e^x - 1)^2}{x^2} = \lim_{x\to 0} \Big(\underbrace{\frac{e^x - 1}{x}}_{\to 1}\Big)^2 = 1^2 = \underline{\underline{1}}$$

c) $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} (e^{\ln x})^x =$

$$= \lim_{x\to 0^+} e^{x \ln x} = e^{\lim_{x\to 0^+} x \ln x} = e^0 = \underline{\underline{1}}$$

e) $\lim\limits_{x\to\infty} x(\ln(2+x) - \ln x) = \lim\limits_{x\to\infty} x \ln \dfrac{2+x}{x} =$

$$= \lim_{x\to\infty} x \ln\left(\frac{2}{x} + 1\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{2}{x}\right)}{\frac{1}{x}} =$$

$$= \lim_{x\to\infty} \frac{2\ln\left(1 + \frac{2}{x}\right)}{\frac{2}{x}} = \underline{\underline{2}}$$

$\to 1$ då $\frac{2}{x} \to 0$

**32** a) $\lim\limits_{x\to\infty} \dfrac{3^x + \ln|x|}{x^5 + x^4} =$ bryter ut dominant term

$x > 0$

$$= \lim_{x\to\infty} \frac{3^x\Big(1 + \underbrace{\frac{\ln x}{3^x}}_{\to 0}\Big)}{x^5\Big(1 + \underbrace{\frac{1}{x}}_{\to 0}\Big)} = \lim_{x\to\infty} \frac{3^x}{x^5} =$$

$$= \frac{1}{\underbrace{\lim\limits_{x\to\infty} \frac{x^5}{3^3}}_{\to 0}} = \underline{\underline{\infty}}$$

b) $$\lim_{x\to 0} \frac{3^x + \ln|x|}{x^5+x^4} = \lim_{x\to 0} \overset{\to\infty}{\left(\frac{1}{x^5+x^4}\right)} \cdot \left(\underset{\to 1}{3^x} + \underset{\to -\infty}{\ln|x|}\right) = \underline{\underline{-\infty}}$$

c) $$\lim_{x\to -\infty} \frac{\overset{\to 0}{3^x} + \overset{\text{dominant}}{\ln|x|}}{\underset{\text{dominant}}{x^5} + x^4} = \lim_{x\to -\infty} \frac{\ln|x|\left(1 + \overset{\to 0}{\frac{3^x}{\ln|x|}}\right)}{x^5\left(1 + \underset{\to 0}{\frac{1}{x}}\right)} =$$

$$\overset{\text{då } x<0}{=} \lim_{x\to -\infty} \frac{\ln(-x)}{x^5} = \left[\begin{array}{l} x = -t \\ x \to -\infty \Leftrightarrow t \to \infty \end{array}\right] =$$

$$= \lim_{t\to\infty} \underset{\to 0}{\frac{\ln t}{-t^5}} = \underline{\underline{0}}$$

**34** a) $$\lim_{x\to\infty} \frac{x^2}{e^{\sqrt{x}}} = \lim_{x\to\infty} \frac{(\sqrt{x})^4}{e^{\sqrt{x}}} = \underline{\underline{0}} \text{ då } \sqrt{x} \to \infty.$$

b) $$\lim_{x\to\infty} \frac{e^x + x\sin x}{2\underset{\text{domin.}}{e^x} - x^2\ln x} = \lim_{x\to\infty} \frac{e^x\left(1 + \frac{x\sin x}{e^x}\right)}{e^x\left(2 - \frac{x^2\ln x}{e^x}\right)} =$$

$$= \lim_{x\to\infty} \left( \frac{1 + \overset{\to 0}{\overset{\to 0}{\frac{x}{e^x}} \cdot \overset{\text{begräns.}}{\sin x}}}{2 - \underset{\to 0}{\frac{x^3}{e^x}} \cdot \underset{\to 0}{\frac{\ln x}{x}}} \right) = \underline{\underline{\frac{1}{2}}}$$

c) $$\lim_{x\to\infty} \left(1 + \frac{1}{5x}\right)^{7x} = \lim_{x\to\infty} \left(1 + \frac{1}{5x}\right)^{5x\cdot\frac{7}{5}}$$

$$= \Bigg(\underset{\to e \text{ då } \frac{1}{5x} \to 0}{\lim_{x\to\infty} \left(1 + \frac{1}{5x}\right)^{5x}}\Bigg)^{\frac{7}{5}} = \underline{\underline{e^{7/5}}}$$

d) $$\lim_{x\to\infty} \frac{x + \ln(e^{2x} + x)}{\sqrt{x} + e^{1+\ln x}} = \lim_{x\to\infty} \frac{x + \ln\left(e^{2x}\left(1 + \frac{x}{e^{2x}}\right)\right)}{\sqrt{x} + e\cdot x}$$

$$= \lim_{x\to\infty} \frac{x + 2x + \ln\left(1 + \frac{x}{e^{2x}}\right)}{x\left(e + 1/\sqrt{x}\right)} =$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\left(3 + \frac{\ln\left(1+\overset{\to 0}{\frac{x}{e^{2x}}}\right)}{x}\right)}{\cancel{x}\left(e + \underset{\to 0}{\frac{1}{\sqrt{x}}}\right)} = \lim_{x\to\infty} \frac{3 + \overset{\to 1}{\frac{\ln\left(1+\frac{x}{e^{2x}}\right)}{\frac{x}{e^{2x}}}} \cdot x \cdot \frac{e^{2x}}{x}}{e}$$

$$= \lim_{x\to\infty} \frac{3 + \overset{\to 0}{\frac{1}{e^{2x}}}}{e} = \frac{3}{e}.$$

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$$\lim_{n\to\infty}\left(\frac{1+2+\dots+n}{n-5} - \frac{n}{2}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{\frac{n(n+1)}{2}}{n-5} - \frac{n}{2}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{n(n+1)}{2(n-5)} - \frac{n}{2}\right) = \lim_{n\to\infty}\frac{n(n+1)-n(n-5)}{2(n-5)} =$$

$$= \lim_{n\to\infty}\frac{n(\cancel{n}+1-\cancel{n}+5)}{2(n+5)} = \lim_{n\to\infty}\frac{n\cdot 6}{(n+5)2} = 3\lim_{n\to\infty}\frac{1}{1+\frac{5}{n}}$$

$$= \underline{\underline{3}}$$

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a) $$\lim_{n\to\infty} n\left(n-\sqrt{n^2-4}\right) =$$

$$= \lim_{n\to\infty}\frac{n\left(n-\sqrt{n^2-4}\right)\left(n+\sqrt{n^2-4}\right)}{n+\sqrt{n^2-4}} =$$

$$= \lim_{n\to\infty}\frac{n\left(\cancel{n^2}-\cancel{n^2}+4\right)}{n+\sqrt{n^2-4}} = \lim_{n\to\infty}\frac{4n}{n+\sqrt{n^2-4}} =$$

$$= \lim_{n\to\infty}\frac{4n}{n+n\sqrt{1-\underset{\to 0}{\frac{4}{n}}}} = \lim_{n\to\infty}\frac{4\cancel{n}}{\cancel{n}\underbrace{\left(1+\sqrt{1-\frac{4}{n}}\right)}_{\to 2}} = 2$$

b) $\lim\limits_{n\to\infty}\left(\dfrac{(n+1)^2}{n-1}-\dfrac{(n-1)^2}{n+1}\right)=\lim\limits_{n\to\infty}\dfrac{(n+1)^3-(n-1)^3}{n^2-1}=$

$=\lim\limits_{n\to\infty}\dfrac{\cancel{n^3}+3n^2+\cancel{3n}+1-\cancel{n^3}+3n^2-\cancel{3n}+1}{n^2-1}=\lim\limits_{n\to\infty}\dfrac{6n^2+2}{n^2-1}=$

$=\lim\limits_{n\to\infty}\dfrac{\cancel{n^2}\left(6+\overset{\to 0}{\dfrac{2}{n^2}}\right)}{\cancel{n^2}\left(1-\underset{\to 0}{\dfrac{1}{n^2}}\right)}=\underline{\underline{6}}.$

c) $\lim\limits_{n\to\infty}\left(1\underset{\to 0}{-\dfrac{1}{2n}}\right)^{n^2}=\lim\limits_{n\to\infty}\underbrace{\left(1-\dfrac{1}{2n}\right)^{-2n}}_{\to e \text{ då } -\frac{1}{2n}\to 0}{}^{\cdot\frac{n^{\cancel{2}}}{-2\cancel{n}}}$

$=\lim\limits_{n\to\infty}e^{-\frac{n}{2}}=\dfrac{1}{\lim\limits_{n\to\infty}\underset{\to\infty}{e^{n/2}}}=\underline{\underline{0.}}$

d) $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{2n}\right)^{n+5}=\lim\limits_{n\to\infty}\underbrace{\left(1-\dfrac{1}{2n}\right)^{-2n}}_{\to e}{}^{\cdot\frac{n+5}{-2n}}=$

$=\lim\limits_{n\to\infty}e^{\frac{n+5}{-2n}}=\lim\limits_{n\to\infty}e^{\frac{\cancel{n}(1+\frac{5}{n})}{\cancel{n}(-2)}}=\lim\limits_{n\to\infty}e^{\frac{1+\frac{5}{n}}{-2}}=$

$=e^{-1/2}=\underline{\underline{\dfrac{1}{\sqrt{e}}}}.$

Extra

$\boxed{B3}$ $\lim\limits_{\lambda\to 0}\dfrac{2\pi hc^2}{\lambda^5\left(e^{\frac{hc}{\lambda kT}}-1\right)}=\lim\limits_{\lambda\to 0}\dfrac{2\pi hc^2}{\lambda^5 e^{\frac{hc}{\lambda kT}}\left(1-\dfrac{1}{e^{hc/\lambda kT}}\right)}$

$=\lim\limits_{\lambda\to 0}\dfrac{\left(\dfrac{hc}{\lambda kT}\right)^5\cdot 2\pi hc^2}{\left(\dfrac{hc}{\cancel{\lambda}kT}\right)^5\cdot\cancel{\lambda^5}\cdot e^{\frac{hc}{\lambda kT}}}=\lim\limits_{\lambda\to 0}\dfrac{2\pi hc^2}{\dfrac{h^5c^5}{k^5T^5}}\cdot 0$

$\to 0$ då $\dfrac{hc}{\lambda kT}\to\infty$ $\quad$ tal

$$= \cancel{2}\pi h \cancel{c^2} \cdot \frac{k^5T^5}{h^{\cancel{5}}_{4} c^{\cancel{5}3}} \cdot 0 = \frac{2\pi k^5 T^5}{h^4 c^3} \cdot 0 = \underline{\underline{0}}.$$

**41** Talföljden $a_n = 1 + x + \dots + x^n$ konvergerar då $\lim_{n\to\infty} a_n$ existerar och ändlig.

Vi använder $1 + x + \dots + x^n = \begin{cases} \dfrac{1-x^{n+1}}{1-x} & \text{då } 0 \le x < 1 \\ n+1 & \text{då } x = 1 \\ \dfrac{x^n - 1}{x-1} & \text{då } x > 1 \end{cases}$

Alltså:

$\underline{0 \le x < 1}$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{1 - \overset{\to 0}{x^{n+1}}}{1-x} = \dfrac{1}{1-x}$

$\underline{x = 1}$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (n+1) = \infty$

$\underline{x > 1}$: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{x^n(1-\frac{1}{x^n})}{x-1} =$

$= \lim_{n\to\infty} \dfrac{x^{n-1}}{x-1} = \infty$

$\underline{-1 < x < 0}$: se $0 \le x < 1$! $\lim_{n\to\infty} a_n = \dfrac{1}{1-x}$.

$\underline{x = -1}$: $\lim_{n\to\infty} a_n$ existerar inte, eftersom

det finns inte $\underline{\text{ett}}$ tal som $a_n$ närmar sig för stora n.

$$\begin{cases} 0 \leftarrow a_{2k} = 1 - 1 + 1 - 1 + \dots + 1 - 1 = 0 \\ 1 \leftarrow a_{2k+1} = 1 - 1 + \dots + 1 - 1 + 1 = 1 \end{cases}$$

$X < -1$: $\lim_{n\to\infty} |a_n| = \dots = \lim_{n\to\infty} \left| \dfrac{x^{n-1}}{x-1} \right| = \infty \Rightarrow$ ingen konvergens.

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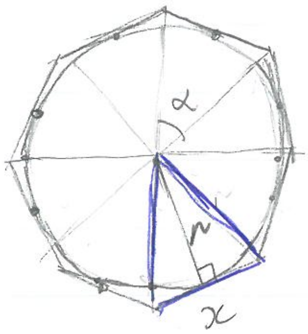
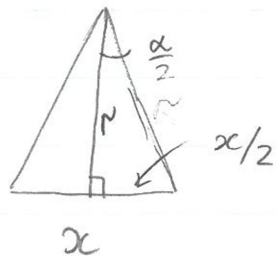
54 Vi bildar en polygon som omsluter en cirkel, har n-sidor, och alla sidor har samma längd. I så fall

rad = r

$\alpha = \frac{2\pi}{n}$. Låt polygons sida vara $x$ $\Rightarrow$




$$\frac{\frac{x}{2}}{r} = \tan\frac{\alpha}{2}$$

$$\Rightarrow x = 2r\tan\frac{\alpha}{2}$$

Vi ser att $A \leq n \cdot \underbrace{\Delta}_{\text{triangelns area}} = n \cdot \frac{1}{2} \cdot x \cdot r =$

$$= n \cdot \frac{1}{\cancel{2}} \cdot \cancel{2}r \cdot \tan\frac{2\pi}{2n} \cdot r = \underbrace{nr^2\tan\frac{\pi}{n}}_{A_n}.$$

När $n \to \infty$: $\lim\limits_{n\to\infty} nr^2\tan\frac{\pi}{n} = \lim\limits_{n\to\infty} r^2 \cdot \underbrace{\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}}_{\to 1} \cdot \frac{1}{\frac{1}{\pi}} =$

$$= \lim_{n\to\infty} \pi r^2 = \pi r^2.$$

Dvs polygonens area närmar sig cirkelns area då $n \to \infty$.