Lektion 14

P5

4 Uppgiften handlar om partiell integration:

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx$$

där $F(x)$ är en primitiv funktion till $f(x)$.

a) $\int x e^{-x}\,dx = \begin{bmatrix} f(x)=e^{-x} & F(x)=-e^{-x} \\ g(x)=x & g'(x)=1 \end{bmatrix} =$

$= -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C.$

b) $\int x^2 \sin 2x\,dx = \begin{bmatrix} f(x)=\sin 2x & F(x)=-\frac{1}{2}\cos 2x \\ g(x)=x^2 & g'(x)=2x \end{bmatrix} =$

$= -\frac{x^2}{2}\cos 2x + \underbrace{\int x\cos 2x\,dx}_{=I} = \otimes$

$I = \int x\cos 2x\,dx = \begin{bmatrix} f(x)=\cos 2x & F(x)=\frac{1}{2}\sin 2x \\ g(x)=x & g'(x)=1 \end{bmatrix} =$

$= \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$

$\otimes = -\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$

c) $x\ln|x| = \begin{bmatrix} f(x)=x & F(x)=\frac{x^2}{2} \\ g(x)=\ln|x| & g'(x)=\frac{1}{x} \end{bmatrix} =$

$= \frac{x^2}{2}\ln|x| - \int \frac{x}{2}\,dx = \frac{x^2}{2}\ln|x| - \frac{x^2}{4} + C$

Först gör variabelbyte

d) $\int (\ln x)^2 dx = \left[\begin{array}{l} \ln x = y \Rightarrow x = e^y \\ dx = d(e^y) = (e^y)' dy = e^y dy \end{array} \right] =$

$= \int y^2 \cdot e^y dy = \left[\begin{array}{ll} f(y) = e^y & F(y) = e^y \\ g(y) = y^2 & g'(y) = 2y \end{array} \right] =$

$= y^2 e^y - \int 2y e^y dy = y^2 e^y - 2 \underbrace{\int y e^y dy}_{=I} = \otimes$

$I = \int y e^y dy = \left[\begin{array}{ll} f(y) = e^y & F(y) = e^y \\ g(y) = y & g'(y) = 1 \end{array} \right] =$

$= y e^y - \int e^y dy = y e^y - e^y + C \Rightarrow$

kan alltid skriva C istället

$\otimes = y^2 e^y - 2y e^y + 2e^y - 2C =$

$= (\ln x)^2 \cdot x - 2 \ln x \cdot x + 2x + C =$

$= x\left((\ln x)^2 - 2\ln x + 2 \right) + C.$

e) $\int x (e^x + \ln x) dx = \underbrace{\int x e^x dx}_{=I_1} + \underbrace{\int x \ln x dx}_{=I_2} = \otimes$

$I_1 = \int x e^x dx = [\text{se } 5.4\ d] = x e^x - e^x + C_1$

$I_2 = \int x \ln x dx = [\text{se } 5.4\ c] = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C_2$

$\otimes = x e^x - e^x + \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$

f) $\int \arctan x \, dx = \left[\begin{matrix} f(x) = 1 & F(x) = x \\ g(x) = \arctan x & g'(x) = \frac{1}{1+x^2} \end{matrix} \right] =$

$= x \arctan x - \underbrace{\int \frac{x}{1+x^2} dx}_{\text{gör t ex variabelbyte } y = 1+x^2 \text{ eller gissa}} = x \arctan x - \frac{1}{2} \ln(1+x^2) + C$

g) $\int \arcsin x \, dx = \left[\begin{matrix} f(x) = 1 & F(x) = x \\ g(x) = \arcsin x & g'(x) = \frac{1}{\sqrt{1-x^2}} \end{matrix} \right] =$

$= x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = \left[\begin{matrix} \left(\sqrt{1-x^2}\right)' = \frac{-2x}{2\sqrt{1-x^2}} \\ \Rightarrow \int -\frac{x \, dx}{\sqrt{1-x^2}} = \sqrt{1-x^2} \end{matrix} \right] =$

$= x \arcsin x + \sqrt{1-x^2} + C$

h) $\int \frac{\sin x \overbrace{\cos x \, dx}^{dy}}{1+\sin^2 x} = \left[\begin{matrix} y = \sin x \\ dy = d(\sin x) = (\sin x)' dx = \cos x \, dx \end{matrix} \right]$

$= \int \frac{y}{1+y^2} dy = \left[\begin{matrix} (\ln(1+y^2))' = \frac{2y}{1+y^2} \\ \Rightarrow \int \frac{y}{1+y^2} dy = \frac{1}{2} \ln(1+y^2) \end{matrix} \right] =$

$= \frac{1}{2} \ln(1+y^2) + C = \frac{1}{2} \ln(1+\sin^2 x) + C$

i) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \left[\begin{matrix} t = \cos x \\ dt = -\sin x \, dx \end{matrix} \right] =$

$= -\int \frac{\overbrace{-\sin x \, dx}^{=dt}}{\cos x} = -\int \frac{dt}{t} = -\ln|t| + C =$

$= -\ln|\cos x| + C.$

6 a) $$\int (x^3+x)e^{x^2}dx = \int (x^2+1)e^{x^2}x\,dx = \left[\begin{matrix} x^2=t \\ dt=2x\,dx \end{matrix}\right]$$

$$= \frac{1}{2}\int (x^2+1)e^{x^2}\underbrace{2x\,dx}_{=dt} = \frac{1}{2}\int (t+1)e^t\,dt =$$

$$= \frac{1}{2}\int te^t\,dt + \frac{1}{2}\int e^t\,dt = \left[\text{se } 5.4\ d\right] =$$

$$= \frac{1}{2}te^t - \cancel{\frac{1}{2}e^t} + \cancel{\frac{1}{2}e^t} + C = \frac{1}{2}x^2e^{x^2} + C$$

b) $$\int x^5\cos x^3\,dx = \int x^3\cos x^3\cdot x^2\,dx =$$

$$= \left[\begin{matrix} x^3=t \\ dt=3x^2\,dx \end{matrix}\right] = \frac{1}{3}\int x^3\cos x^3\cdot\underbrace{3x^2\,dx}_{=dt} =$$

$$= \frac{1}{3}\int t\cos t\,dt = \left[\begin{matrix} f(t)=\cos t & F(t)=\sin t \\ g(t)=t & g'(t)=1 \end{matrix}\right] =$$

$$= \frac{1}{3}\left(t\sin t - \int \sin t\,dt\right) = \frac{1}{3}\left(t\sin t + \cos t\right) + C =$$

$$= \frac{1}{3}\left(x^3\sin x^3 + \cos x^3\right) + C$$

c) $$\int\sqrt{1+\sqrt{x}}\,dx = \left[\begin{matrix} y=1+\sqrt{x} \Rightarrow x=(y-1)^2 \\ dx=\left((y-1)^2\right)'dy=(2y-2)\,dy \end{matrix}\right]$$

$$= \int\sqrt{y}\cdot(2y-2)\,dy = \int 2y^{3/2} - 2y^{1/2}\,dy =$$

$$= \frac{2y^{5/2}}{5/2} - \frac{2y^{3/2}}{3/2} + C = \frac{4y^{5/2}}{5} - \frac{4y^{3/2}}{3} + C =$$

$$= 4y^{3/2}\left(\frac{y}{5} - \frac{1}{3}\right) + C = \frac{4y^{3/2}}{15}(3y-5) + C =$$

$$= \frac{4(1+\sqrt{x})^{3/2}}{15}\left(3\sqrt{x} - 2\right) + C$$

d) $\int e^{\sqrt{x}}\,dx = \left[\begin{array}{ll} \sqrt{x} = y & \Rightarrow x = y^2 \\ & dx = 2y\,dy \end{array}\right] =$

$= \int e^{y} \cdot 2y\,dy = 2\int y e^{y}\,dy = [\text{se } 5.4d] =$

$= 2(ye^{y} - e^{y}) + C = 2(\sqrt{x}\,e^{\sqrt{x}} - e^{\sqrt{x}}) + C =$

$= \underline{e^{\sqrt{x}}(2\sqrt{x} - 2) + C}.$

8 $\int \cos x \sin x\,dx$ är inte bara en primitiv funktion, utan alla primitiva funktioner

Om $f(x)$ är en prim. funktion till $\cos x \sin x$, då är $f(x) + C^{\in \mathbb{R}}$ - alla primitiva funktioner
Vi kan skriva resultatet som

$$\underbrace{f(x) + C_1}_{\text{någon prim. funktion}} = 1 + \underbrace{f(x) + C_2}_{\text{någon prim. funktion}}$$

Vi ser att det inte är någon motsägelse om $C_1 = 1 + C_2$.

9 Söker $f(x) = \int x e^{\sqrt{x}}\,dx$ så att $f(1) = 0$.

$f(x) = \int x e^{\sqrt{x}}\,dx = \left[\begin{array}{ll} \sqrt{x} = y \Rightarrow & x = y^2 \\ & dx = 2y\,dy \end{array}\right] =$

$= \int y^2 \cdot e^{y} \cdot 2y\,dy = 2\int y^3 e^{y}\,dy = \left[\begin{array}{l} f(y) = e^{y},\ F(y) = e^{y} \\ g(y) = y^3,\ g'(y) = 3y^2 \end{array}\right]$

$= 2\left(y^3 e^{y} - 3\int y^2 e^{y}\,dy\right) = 2y^3 e^{y} - 6\underbrace{\int y^2 e^{y}\,dy}_{\text{se } 5.4d} =$

$= 2y^3 e^{y} - 6(y^2 e^{y} - 2ye^{y} + 2e^{y}) + C =$

$$= 2x\sqrt{x}e^{\sqrt{x}} - 6xe^{\sqrt{x}} + 12\sqrt{x}e^{\sqrt{x}} - 12e^{\sqrt{x}} + C =$$

$$= \left(2x\sqrt{x} - 6x + 12\sqrt{x} - 12\right)e^{\sqrt{x}} + C.$$

Eftersom $f(1) = 0 \Rightarrow 0 = (2 - 6 + \cancel{12} - \cancel{12})e + C$

$\Rightarrow C = 4e$, och

$$f(x) = (2x\sqrt{x} - 6x + 12\sqrt{x} - 12)e^{\sqrt{x}} + 4e$$

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8a söker $f(x) = \int e^x \sin x$ så $f(0) = 1$.

$$f(x) = \int e^x \sin x \, dx = \begin{bmatrix} f(x) = e^x & F(x) = e^x \\ g(x) = \sin x & g'(x) = \cos x \end{bmatrix} =$$

$$= e^x \sin x - \underbrace{\int e^x \cos x \, dx}_{= I} = \otimes$$

$$I = \int e^x \cos x \, dx = \begin{bmatrix} f(x) = e^x & F(x) = e^x \\ g(x) = \cos x & g'(x) = -\sin x \end{bmatrix} =$$

$$= e^x \cos x + \int e^x \sin x \, dx \Rightarrow$$

$\otimes = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx + C$ och

$$\int e^x \sin x \, dx = e^x(\sin x - \cos x) - \int e^x \sin x \, dx + C \Rightarrow$$

$$2\int e^x \sin x \, dx = e^x(\sin x - \cos x) + C \Rightarrow$$

$$f(x) = \int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2} + C$$

Eftersom $f(0)=1$,

$$1 = \frac{1\cdot(0-1)}{2} + C \Rightarrow \quad C = \frac{3}{2} \quad \text{och}$$

$$f(x) = \frac{e^x(\sin x - \cos x)}{2} + \frac{3}{2}$$

Extra

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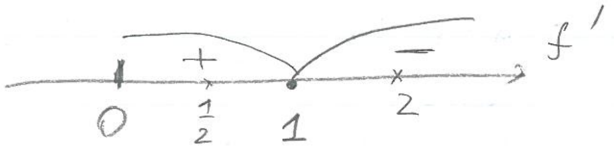
35 Vi vet att $f(x)=\int(1-x^2)e^{-x}dx$, $x\geq 0$, $\lim_{x\to\infty} f(x) = 1$, f_{min} och f_{max} sökes.

$$f'(x) = (1-x^2)e^{-x} = -(x-1)(x+1)e^{-x}, \quad x \geq 0$$



Så $f \uparrow$ då $0 \leq x \leq 1$, 1 är lok. max, och $\lim_{x\to\infty} f(x) = 1$.

För att hitta f_{min} och f_{max} måste vi veta $f(0)$ och $f(1) \Rightarrow$ integralen ska beräknas!

$$f(x) = \int\left(e^{-x} - x^2e^{-x}\right)dx = \left[\begin{array}{l} y=-x, \; x=-y \\ dx = -dy \end{array}\right] =$$

$$= -\int\left(e^y - y^2e^y\right)dy = -\int e^y dy + \int y^2e^y dy =$$

$$= [\text{se } 5.4d] = -e^y + y^2e^y - 2ye^y + 2e^y + C =$$

$$= -e^{-x} + x^2e^{-x} + 2xe^{-x} + 2e^{-x} + C$$

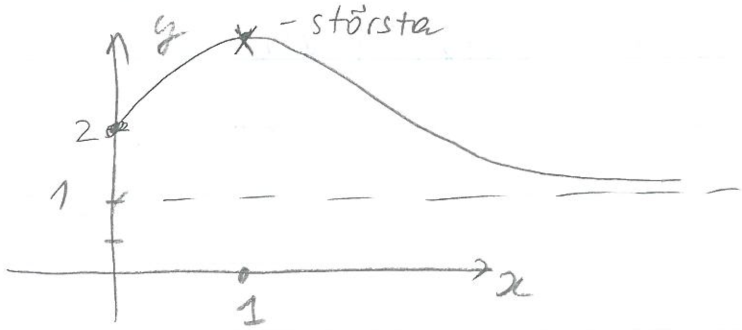
$$= x^2e^{-x} + 2xe^{-x} + e^{-x} + C$$

Eftersom $\lim_{x \to \infty} f(x) = 1 \Rightarrow 1 = \lim_{x\to\infty}\left(\underbrace{\frac{x^2}{e^x} + \frac{2x}{e^x} + \frac{1}{e^x}}_{\to 0} + C\right)$

$\Rightarrow C = 1,$

$f(x) = x^2e^{-x} + 2xe^{-x} + e^{-x} + 1.$

$f(0) = 2, \quad f(1) = 4e^{-1} + 1$



$\Rightarrow$ Största värdet är $1 + \frac{4}{e}$,

Minsta värdet saknas.

<u>36</u> $f(x) = \int x^4 \sin^2 x \, dx \Rightarrow$

$f'(x) = x^4 \sin^2 x \geq 0$, och $=0$ endast i enstaka punkter $x = 0,\ x = \pi n,\ n \in \mathbb{Z}_+$ (så f' är aldrig noll på något intervall).

$\Rightarrow f$ är strängt växande.